

# Superstatistics and the quest of generalized ensembles equivalence in a system with long-range interactions

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## Abstract

The so-called  $\chi^2$ -superstatistics of Beck and Cohen (BC) is employed to investigate the infinite-range Blume-Capel model, a well-known representative system displaying inequivalence of canonical and microcanonical phase diagrams. While not being restricted to any of those particular thermodynamic limits, our analytical result can smoothly recover both canonical and microcanonical ensemble solutions as its nonextensive parameter  $q$  is properly tuned. Additionally, we compare our findings to ones previously obtained from a generalized canonical framework named Extended Gaussian ensemble (EGE). Finally, we show that both EGE and BC solutions are equivalent at the thermodynamic level.

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Key words: superstatistics, nonextensive statistical mechanics, extended gaussian ensemble, Blume-Capel model

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## 1. Introduction

Superstatistics inception by Beck and Cohen [1] was intended to provide an extension of the standard statistical mechanics formalism into a more general one, focusing on describing out-of-equilibrium systems, which are most likely characterized by spatio-temporal fluctuations of an intensive parameter. Its usual formulation [2, 3, 4] employs, as a working hypothesis, the argument that fluctuations evolve on a long-time scale, while the studied system can still be locally decomposed in many small cells (subsystems) obeying the equilibrium statistical mechanics characterized, for instance, by an effective local inverse temperature  $\beta$ . For such systems, not only the temperature environment is considered to be a fluctuating quantity, with probability density  $f(\beta)$ , but also it may carry a spatial modulation as a classical scalar field.

Despite of some early criticism [5], superstatistics has been growing [1, 6, 7, 8, 9] as a consistent framework able to provide deep physical insights for a large variety of complex nonequilibrium stationary systems. It is corroborated by the fact that the understanding of BC formulation of superstatistics greatly profits from the perspective of a Bayesian formalism, as exposed by Sattin [10]. For instance, let us suppose one is interested in measurements of the energy  $E$  emerging as the outcome of an experiment reproducible as many times as desired. The modeling of this experiment depends on some parameters and assumptions related to an underlying stochastic process. So, let  $\beta$  be the parameter that mainly accounts for the dynamics of such experiment, whose due description then implies on a proper knowledge of the parameter. Within the Bayesian framework, the knowledge about  $\beta$  is cast in terms of probabilities known as the

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*prior* distribution  $p(\beta)$  and, the *posterior* distribution described by the conditional form  $p(\beta|E)$  contains all information about  $\beta$  once the observations on  $E$  were given. However, this is clearly an unknown distribution.

In this vein, superstatistics introduces a procedure to obtain marginal probability  $p(E)$  from nonequilibrium dynamical processes once given the prior distribution  $p(\beta)$  [1, 3],

$$p(E) = \int p(E|\beta)p(\beta)d\beta. \quad (1)$$

This circumvents the “learning” from experiment  $p(\beta|E) \sim p(E|\beta)p(\beta)$  [10, 11] by assuming that a prior distribution is known,  $p(\beta) = f(\beta)$ , where  $f(\beta)$  physically accounts for the fluctuating parameter  $\beta$ , whose probability distribution function (PDF) is introduced in an *ad hoc* manner and represents our *degree-of-belief*.

The distribution  $p(E|\beta)$  in the BC (type-B superstatistics [6]) formulation assumes that the thermodynamical description is statistically performed in the canonical ensemble

$$p(E|\beta) = \frac{\rho(E)e^{-\beta E}}{Z(\beta)}, \quad (2)$$

where  $\rho(E)$  is the density of states and  $Z(\beta)$  is the usual normalization constant for a given  $\beta$ . While in the so-called type-A superstatistics formulation  $\tilde{p}(E|\beta) = \rho(E)e^{-\beta E}$  and the normalization of  $p(E)$  in the Eq.(1) is performed *a posteriori*, i.e.  $p(E) = \tilde{p}(E)/Z$ , where  $Z$  is given by

$$Z = \int_E \int \tilde{p}(E|\beta)p(\beta)d\beta dE. \quad (3)$$

Then, every time that an *ansatz* may be assumed from the beginning for  $p(E|\beta)$  the type-B formulation is considered as more convenient. Different functional forms of  $f(\beta)$  have been presented and succeeded in describing many complex physical systems. Among them, we find applications as diverse as in hydrodynamic turbulent flows [12], traffic delays on the British railway network [13], survival-time of cancer patients [14], stock market returns [15] and quark-gluon plasma phenomenology (see [16], and references therein).

The choice of  $f(\beta)$  we will exploit in this study is known as the  $\chi^2$ -distribution

$$f(\beta) = \frac{1}{\beta_0} \frac{c^c}{\Gamma(c)} \left( \frac{\beta}{\beta_0} \right)^{c-1} \exp \left( -\frac{c\beta}{\beta_0} \right), \quad (4)$$

where constants  $\beta_0 > 0$  and,  $n = 2c$  is the number of degrees of freedom of the system. In particular, it deserves to be noted that Eq.(4) may partially recover the so-called Tsallis nonextensive statistics when the constant  $c$  is formally related to the Tsallis parameter  $q$  by  $c = 1/(q - 1)$ . The constant  $\beta_0$  is related to the average and variance of the spatio-temporal fluctuations of physical quantity  $\beta$ , once  $\langle \beta \rangle_f = \int_0^\infty \beta f(\beta) d\beta = \beta_0$  and  $\text{Var}(\beta) = \langle \beta^2 \rangle_f - \beta_0^2 = \beta_0^2/c$ . Still, any coupling of a physical system to finite thermal baths would be expected to be properly described by this somehow interpolating framework, given that  $\beta_0$  can even be identified with a sharply defined inverse temperature in limit when  $\text{Var}(\beta) \rightarrow 0$  i.e., if coupled to a thermal reservoir.

This last property would be specially desirable to better understand the thermal behavior of a large set of systems endowed by long-range interactions [17, 18, 19, 20] and, whose microcanonical and canonical thermodynamical properties present notable inequivalence. It is broadly accepted that those aspects arise as consequences of the nonconcavity of the entropy, seen as a function of the energy [17, 21, 22]. Thus, within an interval  $(\varepsilon_a, \varepsilon_b)$  of energies where the microcanonical entropy is not a concave function, the microcanonical and the canonical ensembles become nonequivalent. A word of caution may be due here, once we distinguish between the microcanonical entropy  $S_\mu(\varepsilon)$  and the canonical one  $S_{can}(\beta)$ , as the latter is obtained as the Legendre-Fenchel transform of the free energy  $\varphi(\beta)$ , an operation that always yields a concave function of  $\beta$ .

To circumvent that technical hindrance, Touchette *et al.* have recently presented ([23], and references therein) a series of methods to analytically calculate entropies that are nonconcave

functions of the energy. This can be implemented by some generalized canonical ensembles [24], as the Gaussian Ensemble [25, 26] or its extended version (EGE) [27], where the variance of temperature is parameterized ( $\gamma$ ) proportionally to the inverse thermal capacity of the heat bath. Therefore, by taking proper limits in this unified approach one can recover usual results in different ensembles [28], even when they are inequivalent in the thermodynamic limit [17].

Thereby, maybe it comes as a startling remark upon aforesaid generalized ensemble approaches, as the  $\chi^2$ -superstatistics and EGE, that their universal thermodynamic equivalence (in the sense of Costeniu [24]) may not be ensured right from the beginning. Then, the quest for the existence of equivalent thermodynamic descriptions of some peculiar physical systems studied under different generalized ensembles can only be set by examining their explicit solutions. This is the purpose of our present article, where a  $\chi^2$ -superstatistics is employed to investigate by explicit calculations the infinite-range Blume-Capel (BEC) model, which is a well-known representative system displaying inequivalence of canonical and microcanonical phase diagrams and, whose EGE solution was lately provided [28].

To this end, the article is organized as follows. A short theoretical review on generalized canonical ensembles is provided in Section 2, where EGE is highlighted as the most eminent representative in a class of interpolating canonical  $\leftrightarrow$  microcanonical ensembles. Section 3 introduces the foundations of a particular limit of superstatistics, whose choice of  $f(\beta)$  is known as  $\chi^2$ -distribution. Here it is emphasized that, most of times, this formulation is equivalent and so allows for recovering physical results derived from the nonextensive Statistical Mechanics of Tsallis [29, 30]. In the Section 4 explicit analytic solutions of the BEC model is provided in the context of  $\chi^2$ -superstatistics. Finally, Section 5 is dedicated to our concluding remarks. For completeness, a sketch of solution for the BEC model is worked out in the canonical ensemble in Appendix A, once its partition function  $Z_{can}$  is widely employed along this article.

## 2. Generalized canonical ensembles

The idea underlying the foundations of generalized ensembles is to enable the computation of the microcanonical entropy  $S_\mu(E)$  [31] from a Legendre transform of a generalized free energy function (for instance, see [23, 24])

$$\varphi_g(\beta) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{N,g}(\beta), \quad (5)$$

whereas use is made of a generalized partition function

$$Z_{N,g}(\beta) = \int e^{-\beta H_N(\sigma) - N g(H_N(\sigma)/N)} d\sigma, \quad (6)$$

formulated with the help of  $g$  as a function of the mean energy per degree of freedom  $H_N(\sigma)/N$ . Then, if a proper choice of  $g$  can be made so implying that  $\varphi_g(\beta)$  is differentiable at  $\beta$ , the microcanonical entropy shall be recovered by taking a generalized Legendre transform

$$S_\mu(E) = \beta E - \varphi_g(\beta) + g(E), \quad (7)$$

where the constraint  $E = \varphi'_g(\beta)$  shall be fulfilled.

In particular, when taking  $g(E) = \tilde{\gamma} E^2/2$  one is straightforwardly led to the simplest form of the so-called Extended Gaussian Ensemble [27], whose partition function  $Z_{N,\tilde{\gamma}}(\beta)$  may be derived from Eq.(6) with the help of some usual Gaussian integrals as

$$e^{-\tilde{\gamma} E^2/2} = \sqrt{\frac{\tilde{\gamma}}{2\pi}} \int_{-\infty}^{\infty} e^{-\tilde{\gamma} x^2/2 - i\tilde{\gamma} E x} dx. \quad (8)$$

Therefore, one can interpret  $Z_{N,\tilde{\gamma}}(\beta)$  as an integral transform [23] of the canonical partition function,

$$Z_{N,\tilde{\gamma}}(\beta) = \sqrt{\frac{\tilde{\gamma} N}{2\pi}} \int_{-\infty}^{\infty} e^{-\tilde{\gamma} N x^2/2} Z_{can}(\beta + i\tilde{\gamma} x) dx. \quad (9)$$

Unfortunately, those integrals are generally not easy to compute for physically sound systems. So, explicit calculations of this kind as performed for the infinite-range BEC model [28], are recognized as quite scarce. In addition, it is illustrative to note that the usual canonical partition function  $Z_{can}(\beta)$  may be trivially recovered by EGE in the  $\tilde{\gamma} \rightarrow 0$  limit, i.e.  $Z_{can}(\beta) = \lim_{\tilde{\gamma} \rightarrow 0} Z_{N, \tilde{\gamma}}(\beta)$ . While due to delta sequence representation  $\delta_{\tilde{\gamma}}(E) = \sqrt{\tilde{\gamma}/\pi} e^{-\tilde{\gamma} E^2}$  one directly recovers  $S_{\mu}(E)$  when  $\tilde{\gamma} \rightarrow \infty$  from aforesaid standard methods.

### 3. Superstatistics: the $\chi^2$ -distribution

Let us define  $U$  as the mean energy of the entire superstatistical system consisting of many cells. Since we are assuming that each cell has an approximately constant inverse temperature  $\beta$ , the energy distribution follows from the usual Boltzmann factor  $e^{-\beta E(\sigma)}$ , where  $E(\sigma)$  is the energy of a given state  $\{\sigma\}$  in the cell. The marginal energy distribution for a system with density of states  $\rho(E)$  becomes

$$p(E; U) \sim \int_0^{\infty} \rho(E) e^{-\beta(E-U)} f(\beta) d\beta. \quad (10)$$

Here, we follow the type-A prescription to obtain the normalization constant, which is now dependent on the constant  $U$ . The particular choice for the probability density  $f(\beta)$  as the  $\chi^2$ -distribution for the fluctuating  $\beta$  with mean value  $\beta_0$  and parameter  $c$  yields the superstatistical version of the Boltzmann-Gibbs (BG) statistics for the physical system. Defining  $\theta = \beta/\beta_0$ , one obtains

$$p(E; U) \sim \int_0^{\infty} \rho(E) w_c(\theta) e^{-\theta \beta_0 (E-U)} d\theta, \quad (11)$$

where the weight function  $w_c(\theta)$  is given by

$$w_c(\theta) = \frac{c^c}{\Gamma(c)} \theta^{c-1} e^{-c\theta}. \quad (12)$$

Then, recalling the following mathematical result

$$\int_0^{\infty} w_c(\theta) e^{-\theta \beta E} d\theta = \left(1 + \frac{\beta E}{c}\right)^{-c}, \quad (13)$$

valid for  $c \geq 1$  and  $c + \beta E > 0$ , the stationary energy distribution becomes

$$p(E; U) = \frac{1}{Z_c(\beta_0, U)} \rho(E) \left[1 + \frac{\beta_0 (E - U)}{c}\right]_+^{-c}, \quad (14)$$

where  $[x]_+ = \max(x, 0)$  and  $Z_c(\beta_0, U)$  is the partition function version of BC  $\chi^2$ -superstatistics,

$$Z_c(\beta_0, U) = \int_{E \in I} \rho(E) \left[1 + \frac{\beta_0 (E - U)}{c}\right]_+^{-c} dE, \quad (15)$$

where  $I$  is the energy range that may be limited by some energy cutoff for confined systems.

At this stage it is tempting to identify  $c$  with the Tsallis parameter  $q$  by  $c = 1/(q - 1)$ , which limits this parameter to  $1 \leq q \leq 2$  in type-A prescription. The value  $q = 2$  is allowed for systems with finite energy range. Thus, the partition function version of the  $\chi^2$ -superstatistics in terms of the parameter  $q$  becomes

$$Z_q(\beta_0, U) = \int_{E \in I} \rho(E) e_q^{-\beta_0 (E - U)} dE, \quad (16)$$

where the notation  $e_q^{-x} = [1 + (q-1)x]^{1/(1-q)}$  for the  $q$ -exponential function is introduced, but needs to be restricted to the above domain in  $q$ .

Such description is analogous to the one where “escort probabilities” are not employed [29]. From a formal point of view, the canonical limit is easily recovered in the  $q \rightarrow 1$  limit:  $e_q(x) \rightarrow \exp(x)$ . This limit describes a system with a very large number of independent degrees of freedom  $n$  and turns the  $\chi^2$ -distribution into a Dirac delta function  $\delta(\beta - \beta_0)$  where  $\beta_0$  has to be defined by a large heat bath which provides the constant temperature. Such prior PDF contrasts, for example, to an uniform prior PDF when little is known about the temperature of the system.

### 3.1. An analytical continuation in $\beta$

Next, we demonstrate how to obtain the complementary values for  $q$  ( $q < 1, q > 2$ ) in superstatistics. However, in this case, one needs to explore the analytical continuation for the inverse of the temperature distribution  $\beta$ :  $f(\beta) \rightarrow f(z)$ . Thus, we start with the contour integral representation for the reciprocal gamma function due to Hankel to obtain the complex normalized version of  $f(\beta)$ ,

$$f(z) = \frac{i}{2\pi\beta_0} \frac{\Gamma(d)}{c^{d-1}} \left(\frac{-z}{\beta_0}\right)^{-d} \exp\left(-\frac{cz}{\beta_0}\right), \quad (17)$$

with  $d \neq 0, -1, -2, \dots$ . The integration in  $z$  needs to be performed along a contour  $\Gamma$  starting at  $+\infty$  on the real axis, going counterclockwise around the origin and back to  $+\infty$  [32]. This derivation is similar to real  $\beta$  case when one starts from the normalized gamma function:  $\frac{1}{\Gamma(c)} \int_0^\infty t^{c-1} e^{-t} dt = 1$ . Therefore, the corresponding real partition function becomes

$$Z_d(\beta_0, U) = \int_{E \in I} \int_{\Gamma} \rho(E) \omega_d(\theta) e^{-\theta\beta_0(E-U)} d\theta dE, \quad (18)$$

where  $\theta = z/\beta_0$  is the new complex integration variable and

$$\omega_d(\theta) = \frac{i}{2\pi} \frac{\Gamma(d)}{c^{d-1}} (-\theta)^{-d} e^{-c\theta}. \quad (19)$$

The integration over the contour  $\Gamma$  is easily performed and one obtains

$$Z_d(\beta_0, U) = \int_{E \in I} \rho(E) \left[1 + \frac{\beta_0(E-U)}{c}\right]_+^{d-1} dE. \quad (20)$$

The identifications  $c = 1/(q-1)$  and  $d-1 = 1/(1-q)$  yield the known results of the  $q$ -statistics but now the parameter  $q$  is restricted to  $q < 1$  and  $q > 2$ . This formal construction yields the complementary range of validity in the values of  $q$  for the  $\chi^2$ -superstatistics. It is quite similar to the integral parameterization of the Tsallis partition function  $Z_q$  used by Prato [33] to extend the so-called Hilhorst formula ( $q > 1$ ) to  $q < 1$ .

## 4. Beck-Cohen $\chi^2$ -superstatistics solutions of BEC model

The superstatistics formulation for nonequilibrium systems described by the fluctuation parameter  $\beta$  according to the  $\chi^2$ -distribution has energy distribution given by Eq. (11), which can be cast in the form of the following integral transform

$$p(E; U) \sim \int_0^\infty w_c(\theta) e^{\theta\beta_0 U} Z_{can}(\theta\beta_0) d\theta. \quad (21)$$

The BEC model is described by  $Z_{can}(\beta)$  given in Eq. (A.4) for a system with  $N$  spins and average energy  $U$ . The thermodynamic free energy density version of BC superstatistics is defined as

$$\varphi_c(\beta_0, \varepsilon) = - \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_c(\beta_0, \varepsilon), \quad (22)$$

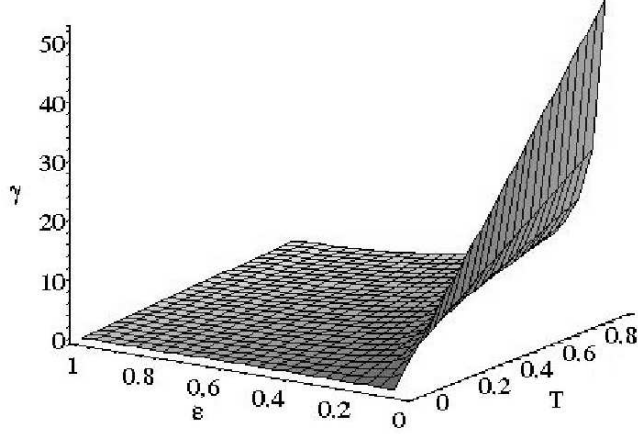


FIG. 1: Maximum values allowed for  $\gamma$  according to Eq. (25) as a function of Energy ( $\varepsilon$ ) and Temperature ( $T$ ) when  $\Delta/J = 0.462407$ .

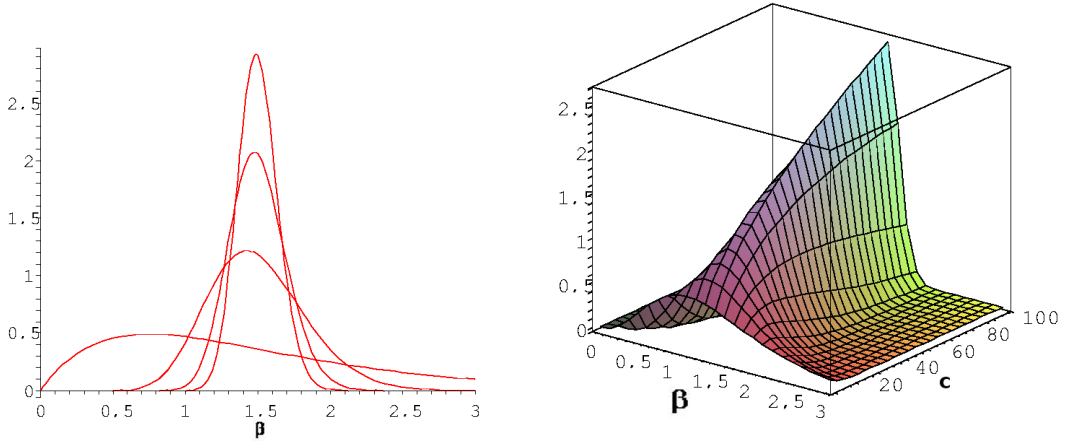


FIG. 2: LEFT PANEL: The  $\chi^2$ -distribution as given by Eq.(4) as a function of  $\beta$  and increasing values of  $c = \{2, 20, 60, 120\}$  for fixed  $\beta_0 = 3/2$ . RIGHT PANEL: the same distribution but in a 3d perspective. Note that when  $c \rightarrow \infty$  a Dirac delta distribution is recovered.

where  $Z_c(\beta_0, \varepsilon)$  is the normalization factor in Eq. (16) written as a function of the average energy per spin  $\varepsilon = U/N\Delta$ .

As usual, the partition function  $Z_c(\beta_0, \varepsilon)$  can be evaluated in the large  $N$  limit by means of the saddle-point approximation, as shown in Eq. (A.5) for the canonical ensemble. Afterwards, the integration over  $\theta$  is analytically performed, which introduces an extra factor that depends on  $\varepsilon$  and  $c$ :  $(1 - \beta_0 \varepsilon N \Delta / c)^c$ . Thus, in terms of  $c = 1/(q - 1)$  and a newly introduced constant  $\gamma = N(q - 1) = N/c$ , the free energy density is given by

$$\begin{aligned} \varphi_\gamma^*(\beta_0, \varepsilon, m, p) = & \frac{1}{\gamma} \ln [1 + \gamma \beta_0 \Delta (p - K m^2 - \varepsilon)] - \frac{1}{\gamma} \ln (1 - \gamma \beta_0 \Delta \varepsilon) \\ & + \ln \left\{ \left[ \frac{\sqrt{p^2 - m^2}}{2(1-p)} \right]^p \left[ \frac{p+m}{p-m} \right]^{m/2} (1-p) \right\}. \end{aligned} \quad (23)$$

The notation  $\varphi^*$  means that  $\varphi$  is evaluated at its stationary points  $(m^*, p^*)$ , whereas the variables  $\varepsilon$ ,  $m$  and  $p$  are considered to be independent. While the third term of Eq.(23) is straightforwardly expressed as  $S_\mu(\varepsilon)/N$  [18, 28], the microcanonical constraint  $\varepsilon = p - K m^2$  is never enforced in

this approach. Still, it is worthy to note that the integration over  $\theta$  leads to the constraints

$$-1 < \gamma\beta_0\Delta(p - Km^2 - \varepsilon) \quad (24)$$

$$\gamma\beta_0\Delta\varepsilon < 1, \quad (25)$$

which must be satisfied together with other saddle point conditions.

Figure 1 shows the maximum values of  $\gamma$  that satisfies the condition (25) as a function of  $\varepsilon$  and  $T_0 = 1/\beta_0$ . In particular, once the BG statistics exhibits a first-order phase transition at ( $T \simeq 0.330666, \varepsilon \simeq 0.330$ , see [18, 28]) for the coupling  $\Delta/J = 0.462407$ , one shall not exceed  $\gamma \sim 2.17$  if it is expected to recover the BG thermodynamic results in the large  $N$  limit, given that  $f(\beta) \rightarrow \delta(\beta - \beta_0)$  as a consequence of  $Var(\beta) \rightarrow 0$  when  $N \rightarrow \infty$ . This limiting behavior is followed by the mode of the distribution tending to  $\langle \beta \rangle_f$  because the maximum of  $f(\beta)$  occurs at  $\beta = \beta_0(1 - 2/N)$ , as seen in Figure 2.

By keeping  $\gamma$  constant  $\varphi^*$  is independent of  $N$  and in this case as the number of spins increases  $q$  goes asymptotically to 1. Accordingly, for a finite and small  $\gamma$  one is allowed to Taylor expand Eq. (23) as a functions of that parameter,

$$\begin{aligned} \varphi_\gamma^*(\beta_0, \varepsilon, m, p) &= \beta_0\Delta(p - Km^2) + \frac{\gamma}{2}\beta_0^2\Delta^2[\varepsilon^2 - (p - Km^2 - \varepsilon)^2] \\ &\quad + \frac{\gamma^2}{3}\beta_0^3\Delta^3[\varepsilon^3 + (p - Km^2 - \varepsilon)^3] + \mathcal{O}(\gamma^3) \\ &\quad + \ln \left\{ \left[ \frac{\sqrt{p^2 - m^2}}{2(1-p)} \right]^p \left[ \frac{p+m}{p-m} \right]^{m/2} (1-p) \right\}. \end{aligned} \quad (26)$$

Surprisingly, up to its lowest orders the expansion in Eq. (26) closely resembles the exact solution formerly obtained in the EGE framework [28], where the free energy was given by

$$\begin{aligned} \varphi_{\tilde{\gamma}}(\beta_0, \varepsilon, m, p) &= \beta_0\Delta(p - Km^2) + \tilde{\gamma}\Delta^2(p - Km^2 - \varepsilon)^2 + \\ &\quad \ln \left\{ \left[ \frac{\sqrt{p^2 - m^2}}{2(1-p)} \right]^p \left[ \frac{p+m}{p-m} \right]^{m/2} (1-p) \right\}. \end{aligned} \quad (27)$$

There  $\tilde{\gamma}$  denotes the EGE free-parameter once explicit in Eq.(9) and, by whose tuning it was shown that the extremum of  $\varphi_{\tilde{\gamma}}(\beta_0, \varepsilon, m, p)$  was able to interpolate between solutions in the canonical ( $\tilde{\gamma} = 0$ ) and microcanonical ensembles ( $\tilde{\gamma} \rightarrow \infty$ ). Thus, by studying the analytic solutions dependent on  $\gamma$  (and  $\tilde{\gamma}$ ) we may infer that  $\chi^2$ -superstatistics and EGE approaches would perhaps produce equivalent thermodynamic descriptions for the BEC model<sup>1</sup>. This hypothesis was earlier proposed in a broader and more abstract perspective by Johal [27] and Morishita [34], but till now it has been waiting to be corroborated by an explicit calculation.

Therefore, to investigate the full thermodynamics of BEC model in our BC approach, we have numerically studied the behaviour of  $\varphi_\gamma^*(\beta_0, \varepsilon, m, p)$  at the points  $m^*$  and  $p^*$  that minimize  $\varphi$  as a function of parameters  $\beta_0$  and  $\varepsilon$  for a set of  $\gamma$  values. We observe that the limiting case  $\gamma \rightarrow 0$  trivially yields the canonical ensemble results (A.6). Additionally, it was verified how large the parameter  $\gamma$  might be set, while obeying the constraints from Eqs. (24) and (25), to possibly recover the well-known microcanonical limit. A summary of our studies is depicted in Figure 3, where the caloric curves  $T(\varepsilon) \times \varepsilon$  extracted from EGE and  $\chi^2$ -superstatistics solutions are plotted for various values of  $\gamma$  (or, respectively  $\tilde{\gamma}$ ) for comparative purposes. We observe that in contradistinction to EGE paradigm where microcanonical results would be recovered as a limiting case  $\tilde{\gamma} \rightarrow \infty$ , the  $\chi^2$ -superstatistics had not to rely on analogous formal limits and, in fact, it was able to fully generate the microcanonical regime for quite small values of  $\gamma$ . So, we verify that  $\chi^2$ -superstatistics and EGE can be considered as successful interpolating approaches for Statistical Mechanics which present ensemble equivalence at the thermodynamic level.

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<sup>1</sup>It is in principle expected that for “small” values of  $\gamma$  and  $\tilde{\gamma}$  the Eq. (26) and (27) becomes equivalent till second order. But, in fact, it comes as a surprise that the thermodynamics emerging from both solutions is the same even when  $\tilde{\gamma} \rightarrow \infty$  but while  $\gamma \sim 0$ .

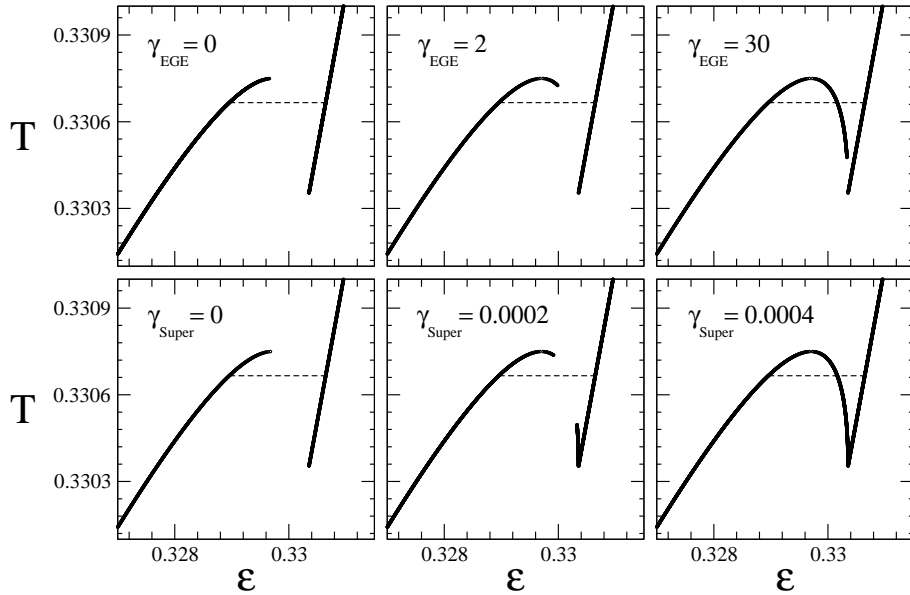


FIG. 3: Caloric curves  $T(\varepsilon) \times \varepsilon$  emerging from the minimization of EGE (upper) and  $\chi^2$ -superstatistics (lower) solutions, depicted respectively from Eqs. (27) and (23). The BEC coupling is set to  $\Delta/J = 0.462407$ , which implies on a first order phase transition in the canonical ensemble with a critical temperature  $T \simeq 0.330666$ , seen as dashed lines (Maxwell construction). By increasing values of parameters for EGE ensemble ( $\gamma_{EGE}$ ) and  $\chi^2$ -superstatistics ( $\gamma_{Super}$ ) the microcanonical caloric curve is gradually recovered, so corroborating the thermodynamic equivalence of both ensembles.

## 5. Concluding remarks

We have evaluated the free energy for the infinite-range BEC model, a system of spins that does not present equivalence between usual (i.e. canonical and microcanonical) ensembles of Statistical Mechanics by employing two generalized statistical frameworks, to know, the  $\chi^2$ -superstatistics and EGE.

Both generalized ensembles here employed were primarily devised to describe systems in contact with finite thermal baths, so presenting peculiar kinds of stationary thermal behavior. Then, thermal fluctuations on those heat baths can be addressed within the superstatistics approach by assuming a probability distribution function  $f(\beta)$ , which we take as being a  $\chi^2$ -distribution, inspired by the fact that this distribution reproduces thermodynamic results resembling the well-known Tsallis nonextensive statistics for various regimes of  $q$ . On the other hand, the EGE approach is based on the assumption that gaussian thermal fluctuations happen in the vicinity of thermal equilibrium, once it is established by physical systems coupled to finite reservoirs. This was proved to consist on a powerful tool for studying systems with nonconcave entropies as proposed in Ref. [23].

Actually, it would be reasonable to expect that depending on the physical system studied by both aforesaid statistical frameworks the resulting thermodynamic predictions might even differ, at least in specific regimes [34], as supposedly seen from some experiments on hadronic production processes[35]. However, it was amazing to find that both our exact solutions for the infinite-range BEC model, obtained in the EGE and  $\chi^2$ -superstatistics, were able to provide us with the same thermodynamical outcome in all conceivable physical regimes described by tuning  $\gamma$  or  $\tilde{\gamma}$ . That is, when  $\gamma = \tilde{\gamma} = 0$  the usual canonical solution was restored as a limiting case by both generalized ensembles, while by taking  $\tilde{\gamma} \rightarrow \infty$  the EGE solution converged to the microcanonical case: a regime that  $\chi^2$ -superstatistics was also able to reach, but already at very small values of  $\gamma$ .

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## Appendix A. The Blume-Capel model in the canonical ensemble

Let us consider the mean-field version of the BEC model for  $N$  spins  $S_i$ ,

$$E(S) = \Delta \sum_{i=1}^N S_i^2 - \frac{J}{2N} \left( \sum_{i=1}^N S_i \right)^2, \quad (\text{A.1})$$

where  $S_i = \{0, \pm 1\}$ . The couplings  $J > 0$  and  $\Delta$  are the exchange and crystal-field interactions, respectively. The BEC model represents a simple generalization of the spin-1/2 Ising model, but with a rich phase diagram in the  $(\Delta/J, T/J)$  plane. It exhibits a first-order transition line, tricritical point, and a second-order transition line.

It is useful to introduce the order parameters magnetization  $M = \sum_{i=1}^N S_i = N_+ - N_-$  and its second moment, the quadrupole moment  $P = \sum_{i=1}^N S_i^2 = N_+ + N_-$ , where  $N_+$  and  $N_-$  are, respectively, the number of sites with up and down spins to describe this model. If  $N_0$  is defined as the total number of zero spins, then  $N = N_+ + N_- + N_0$  is the total number of spins in the system, then Eq. (A.1) is rewritten as

$$E = \Delta P - \frac{J}{2N} M^2. \quad (\text{A.2})$$

After summing up over all configuration states<sup>2</sup>, we arrive to the following formal canonical partition function described in terms of the conditioned order parameters  $P$  and  $M$  instead of the number of spins  $N, N_+$  and  $N_-$ ,

$$Z_{can}(\beta) = \sum_{P=0}^N \sum_{M=-P}^P \frac{N!}{(N-P)! \left[\frac{1}{2}(P-M)\right]! \left[\frac{1}{2}(P+M)\right]!} e^{-\beta E(M,P,N)} \quad (\text{A.3})$$

$$\equiv \sum_{P=0}^N \sum_{M=-P}^P \Omega(M, P, N) e^{-\beta E(M,P,N)}. \quad (\text{A.4})$$

As usual,  $\beta$  stands for the inverse of thermodynamic temperature and we take the Boltzmann constant  $k_B = 1$ .

To evaluate the above expression for large  $N$  limit, we consider the variational approach (saddle-point approximation), which turns the double sum over configurations of  $Z_{can}$  into

$$Z_{can}(\beta) \simeq e^{-N\varphi^*(\beta, m, p)}, \quad (\text{A.5})$$

where  $\varphi^*$  means that the free energy is asymptotically evaluated for large  $N$  around the stationary points  $m^*$  and  $p^*$ . For such calculation, it is convenient to work with intensive quantities  $m = M/N$  and  $p = P/N$ , and define  $K = J/(2\Delta)$ . Therefore, performing the expansion around the stationary point up to second order [31] and using the Stirling's approximation  $\ln N! \simeq N \ln N - N$ , one obtains the free energy,

$$\varphi^*(\beta, m, p) = \beta \Delta (p - K m^2) + \ln \left\{ \left[ \frac{\sqrt{p^2 - m^2}}{2(1-p)} \right]^p \left[ \frac{p+m}{p-m} \right]^{m/2} (1-p) \right\}, \quad (\text{A.6})$$

where the notation  $\varphi^*$  also implies that the points  $m$  and  $p$  are solutions of the minimization equations

$$\frac{\partial \varphi}{\partial m} = \frac{\partial \varphi}{\partial p} = 0, \quad (\text{A.7})$$

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<sup>2</sup>It is convenient to obtain the canonical partition function  $Z_{can}$  of this model by taking the limit  $\gamma \rightarrow 0$  in [28], where  $\gamma$  can be considered just as a parameter to perform an integral regularization procedure [36].

as well as must satisfy the stability condition

$$\det \begin{pmatrix} \frac{\partial^2 \varphi}{\partial m^2} & \frac{\partial^2 \varphi}{\partial m \partial p} \\ \frac{\partial^2 \varphi}{\partial p \partial m} & \frac{\partial^2 \varphi}{\partial p^2} \end{pmatrix} \geq 0, \quad (\text{A.8})$$

for fixed energies  $\varepsilon = E/\Delta N$ . Equations (A.7) and (A.8) is all one needs to describe the thermodynamic behavior of the BEC model. We also emphasizes that  $p$  and  $m$  are independent variables. Thus, the constraint  $\varepsilon = p - Km^2$  is not enforced, contrary to the thermodynamic description in the microcanonical approach.

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